MATH185 Notes

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Introduction

The integers provide an inherent improvement on the naturals by providing the additive inverse. Likewise, the rationals provide the multiplicative inverse and the reals provide completeness. The complex numbers offer the additional advantage in that they are algebraically closed, in that $\mathbb{C}$ contains the solution to every polynomial. This property is the key motivation for the study of complex analysis.

In this course we primarily cover functions from $\mathbb{C} \to \mathbb{C}$ paired with the idea of being holomorphic, the complex analog to being differentiable. The key difference here is the lack of a total ordering as in the reals. We study holomorphic functions using the contour integral, regularity (infinite differentiability), and analytic continuation.

Contents

1 Preliminaries

1.1 Complex Numbers and the Complex Plane

This is a very basic review of what complex numbers are and their properties.

1.1.1 Basic Properties

A **complex number** takes the form $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. We call x the **real part** and y the **imaginary part** of z, written as $x = \mathfrak{Re}(z)$ and $y = \mathfrak{Im}(z)$. The field of complex numbers is denoted with the symbol $\mathbb{C}$. We visualize the complex numbers with a mapping $\mathbb{C} \to \mathbb{R}^2$, where $x + iy$ which is equivalent to the point (x, y). In this analog, we call the x-axis the **real axis** and the y-axis the **imaginary axis**. Complex numbers are a field, and therefore have the commutative, associative, and distributive properties for addition and multiplication – although it should be noted that the exact *meaning* of addition and multiplication is not the same as it is for the reals.

The notion of length is the same as in Euclidian space, in that $|z| = (x^2 + y^2)^{1/2}$ (which makes sense with our relation to $\mathbb{R}^2$). We additionally adhere to the triangle inequality with, for two complex numbers z and w, that

$$||z| - |w|| \leq |z - w|.$$

The **conjugate** of a complex number z is denoted $\bar{z}$, with $\bar{z} = x - iy$, and represents a reflection across the real axis. In turn we get the following:

$$\mathfrak{Re}(z) = \frac{z + \bar{z}}{2}, \quad \mathfrak{Im}(z) = \frac{z - \bar{z}}{2i}, \quad |z|^2 = z\bar{z}, \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Any non-zero complex number can be written as $z = re^{i\theta}$, called **polar form**. θ is the **argument**, and is sometimes denoted $\arg z$. Also recall **Euler's formula**, that

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

This introduction of polar form is why we mentioned earlier that the "meaning" of multiplication is not the same as it is for reals; note that the product of $z, w \in \mathbb{C} = zw = (re^{i\theta})(se^{i\varphi}) = rse^{i(\theta+\varphi)}$ is a rotation combined with a dilation (a **homothety** in $\mathbb{R}^2$).

1.1.2 Convergence

Convergence

A **sequence** $\{z_n\}_{n=1}^{\infty}$ is said to **converge** to $w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0.$$

An interesting addition is that $z_n \to w$ if and only if $\mathfrak{Re}(z_n) \to \mathfrak{Re}(w)$ and $\mathfrak{Im}(z_n) \to \mathfrak{Im}(w)$. An equivalent condition is that the sequence is **Cauchy**; that is, for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n, m > N$ we have $|z_n - z_m| < \varepsilon$. A consequence of the convergence of the real and imaginary parts of a complex sequence is that the complex numbers are necessarily **complete**.

1.1.3 Topology Review in the Complex Plane

This is a brief review of topological concepts from Math104, applied to the complex plane (which is again identical to $\mathbb{R}^2$). For $z_0 \in \mathbb{C}$ and $r > 0$, the **open disc** $D_r(z_0)$ **of radius** r **about** z_0 is the set of all complex numbers within distance r of z_0:

$$D_r(z_0) = \{z \in \mathbb{C} : |z_n - z_0| < r\}.$$

The **closed disc** changes the $<$ to a $\leq$ and is denoted $\bar{D}_r(z_0)$. The boundary of either the open or closed disc is the **circle** $C_r(z_0) = \{z \in \mathbb{C} : |z_n - z_0| = r\}$. Given $\Omega \subset \mathbb{C}$, z_0 is an **interior point** of Ω if there exists an open disc with $r > 0$ contained in Ω. The set of all interior points is the **interior**; a set consisting exclusively of interior points is **open**. The complement of an open set is **closed**. A limit point of a set is a point z such that there exists a sequence $z_n \neq z$ in Ω which converges to z. Closed sets must contain all their limit points. The union of Ω and its limit points is the **closure** of Ω, denoted $\bar{\Omega}$. The boundary $\partial\Omega$ is the closure minus the interior.

A set Ω is **bounded** if there is an $M > 0$ such that $|z| < M$ for every $z \in \Omega$. If Ω is bounded, its **diameter** is

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|.$$

A closed, bounded set is **compact**. Just as with the reals, a set $\Omega \subset \mathbb{C}$ is compact if and only if ever sequence of complex numbers in Ω has a subsequence that converges in Ω.

An **open covering** is a family of open sets $\{U_\alpha\}$ such that

$$\Omega \subset \bigcup U_\alpha.$$

As with the reals, a set is compact if and only if every open cover has a finite subcover.

We also introduce the **nested set property** – that for a sequence of nonempty compact sets $\Omega_1 \supset ...$ of decreasing diameter, there exists some $w \in \Omega_n$ for every n.

Finally, an open set is **connected** if it is not possible to find two disjoint, nonempty open sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \Omega$.

1.2 Functions on the Complex Plane

1.2.1 Continuous Functions

Continuity

Let f be defined on $\Omega \subset \mathbb{C}$. f is **continuous at** $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$. An equivalent definition is that for every $\{z_n\} \to z_0$, $f(z_n) \to f(z_0)$. f is continuous if it is continuous at every $z \in \Omega$.

if f is continuous then the real-valued function $z \to |f(z)|$ is also continuous. f attains a **maximum** at the point $z_0 \in \Omega$ if, for all $z \in \Omega$, $|f(z)| \leq |f(z_0)|$.

Extreme Value Theorem

A continuous function on a compact set Ω is bounded and attains a maximum and minimum on Ω.

1.2.2 Holomorphic Functions

Here we introduce the concept of the **holomorphic function**, the term for complex differentiability.

Holomorphic Functions

Let Ω be open in $\mathbb{C}$ and f a complex-valued function on Ω. The function f is **holomorphic at** $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit as $h \to 0$. This limit, if it exists, is written $f'(z_0)$ and is called the **derivative** of f at z_0. Since h is complex, it may converge to 0 from *any* direction.

f is **holomorphic on** Ω if f is holomorphic at every $z \in \Omega$. If $C \subset \mathbb{C}$ is a closed set, then f is holomorphic on C if f is holomorphic on $\Omega \supset C$, $\Omega \subset \mathbb{C}$. If f is holomorphic on all of $\mathbb{C}$ then we say f is **entire**. We sometimes refer to holomorphic functions as **analytic**, since it is true that every holomorphic function has a power series expansion near every point.

Here are some key properties of holomorphic functions; assume f and g are holomorphic in Ω.

(i) $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$;

(ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$;

(iii) If $g(z_0) \neq 0$ then f/g is holomorphic in Ω and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

(iv) If $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, then the chain rule holds:

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Real and complex differentiability differ significantly. The map $f(z) = \overline{z}$ corresponds to the map in $\mathbb{R}^2$ $F : (x, y) \mapsto (x, -y)$, which is real-differentiable. However, we see that $f(z) = \overline{z}$ is *not* holomorphic, since

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\overline{h}}{h}$$

which has no limit as $h \to 0$; if h is real then the limit is 1, andd if h is pure imaginary then the limit is -1.

Instead, we relate to each complex function $f : u+iv$ the mapping $F(x, y) = (u(x, y), v(x, y))$ from $\mathbb{R}^2$ to R^2. This is then differentiable via the Jacobian:

$$J = J_F(x, y) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}.$$

Cauchy-Riemman Equations

Using our definition of holomorphism, and having $h \to 0$ with h purely real and h pure imaginary, we see that in the first case $f'(z_0) = \frac{\partial f}{\partial x}(z_0)$ and in the latter $f'(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0)$. We can find the partial derivatives with respect to u and v by separating the real and imaginary parts of $f = u + iv$, yielding the following nontrivial relations (the **Cauchy-Riemann equations**):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f/\partial z$.

If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

If we write $F(x, y) = f(z)$, then F is differentiable in the sense of the reals and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

1.2.3 Harmonic Functions

The **Laplacian** operator Δ is defined as

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A function f such that $\Delta f = 0$ is **harmonic**. If f is holomorphic in the open set Ω, then the real and imaginary parts of f are harmonic.

1.2.4 Power Series

Consider the power series for the complex exponential function,

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For $z \in \mathbb{R}$, this is precisely the same as the normal real exponential; furthermore this series converges absolutely for every $z \in \mathbb{C}$.

In general, a **power series** is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

$a_n \in \mathbb{C}$. We test for absolute convergence by examining

$$\sum_{n=0}^{\infty} |a_n||z|^n,$$

noting that if the series converges for some z_0 it converges for all z in the disc $|z| \leq |z_0|$.

Radius of Convergence

Given a power series $\sum a_n z^n$, there exists some $0 \leq R \leq \infty$ such that the series **converges absolutely** if $|z| < R$, and **diverges** if $|z| > R$. R is the **radius of convergence** and $|z|$ is the **disc of convergence**. R is, as in real analysis, given by **Hadamard's formula**

$$1/R = \limsup |a_n|^{1/n}.$$

Further examples include the sin and cos functions, which are analogous to their real counterparts:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

Some substitution simplifies these to

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz}z - e^{-iz}}{2},$$

known as the **Euler formulas**.

The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function within its disc of convergence. We can find the derivative on this disc by differentiating each term of the series individually:

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Furthermore, the radius of convergence of f' is the same as the radius of convergence of f. As a result, a power series is infinitely complex differentiable within its disc of convergence.

Note that we are also allowed to *translate* the power series, i.e. we may have a power series at a point z_0, at which point all the above still hold, substituting $(z - z_0)$ for z (i.e. $f(z) = \sum a_n(z - z_0)^n$). A function on an open set Ω is **analytic** if there exists a power series centered at z_0 with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all z in the neighborhood of z_0.

1.3 Integration along Curves

Just like in metric differential geometry (see my Math140 notes), we have to distinguish between the **parametrization** of a curve (the mapping $[a, b] \to \mathbb{C}$ which is non-unique) and its **geometry** (the unique object (with orientation) in the plane)). A **parameterized curve** is a function $z(t)$ mapping a closed interval to the complex plane. A parameterized curve is **smooth** if $z'(t)$ exists and is continuous on $[a, b]$, and $z'(t) \neq 0$ for $t \in [a, b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as the one-sided limits

$$z'(a) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{z(a + h) - z(a)}{z(h)}, \quad z'(b) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{z(a + h) - z(a)}{h}.$$

A curve is **piecewise smooth** if it is smooth on each interval in a partition of $[a, b]$. Two parametrizations z and $\tilde{z}$ are **equivalent** if there exists a continuous bijection $s \mapsto t(s)$ so that $t'(s) > 0$ and $\tilde{z}(s) = z(t(s))$. The family of all parametrizations equivalent to $z(t)$ is a **smooth curve** $\gamma \subset \mathbb{C}$ (γ^- is γ with orientation reversed). A piecewise smooth curve is defined analogously. The curve has **endpoints** $z(a)$ and $z(b)$, and is called **closed** if $z(a) = z(b)$. If the curve is not self-intersecting, we call it **simple**. We call any piecewise smooth curve a "curve" for brevity.

As an example, let's look at the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The **positive orientation** (counterclockwise) is the standard parametrization

$$z(t) = z_0 + re^{it}$$

for $t \in [0, 2\pi]$; the **negative orientation** is $z(t) = z_0 + re^{-it}$.

We now turn our attention to integration.

Integration

Given a smooth curve γ in $\mathbb{C}$ parameterized by $z : [a, b] \to \mathbb{C}$, and f, a continuous function on γ, the **integral of f along** γ is

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

This definition is independent of the parameterization of γ.

If γ is piecewise smooth, then we sum the integrals over the smooth parts of γ:

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt.$$

The **length** of the smooth curve is

$$\text{length}(\gamma) = \int_\gamma |z'(t)|dt.$$

We then have the following properties:

(i) Integration is **linear**, meaning

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz.$$

(ii) If γ^- is the reverse of γ:

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

(iii) The following inequality holds:

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma).$$

A **primitive** for f on an open set Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$. From this we get a kind of analog to the Fundamental Theorem of Calculus; namely that if a continuous function has primitive F in Ω and begins at w_1 and ends at w_2, and γ is a curve in Ω, then

$$\int_\gamma f(z)dz = F(w_2) - F(w_1).$$

We also introduce **Cauchy's theorem**, the main focus of the next chapter; namely, if γ is a *closed* curve in an open set Ω, and f is continuous with a primitive in Ω, then

$$\oint_\gamma f(z)dz = 0.$$

As a final remark, if f is holomorphic in Ω and $f' = 0$, then f is a constant.

2 Cauchy's Theorem and its Applications

This section deals with one of the most fundamental results in complex analysis, mentioned at the end of the last section.

Cauchy's Theorem

If f is holomorphic in an open set Ω, and $\gamma \subset \Omega$ is a closed curve whose interior is contained in Ω, then

$$\oint_\gamma f(z)dz = 0.$$

A rigorous explanation for what this means (i.e. what it means to be on the "interior" of the curve) is left for later in study; in this section, we instead use **toy contours**, which are contours whose geometries are so straightforward and unambiguous to where Cauchy's theorem can be proved directly.

2.1 Goursat's Theorem

Goursat's Theorem

If Ω is an open set in $\mathbb{C}$, and $\Delta \subset \Omega$ a triangle whose interior is also in Ω, then

$$\oint_{\Delta} f(z)dz = 0$$

whenever f is holomorphic in Ω.

As a corollary, the result is the same if we instead integrate over a rectangle $R \subset \Omega$.

2.2 Local Existence of Primitives and Cauchy's Theorem in a Disc

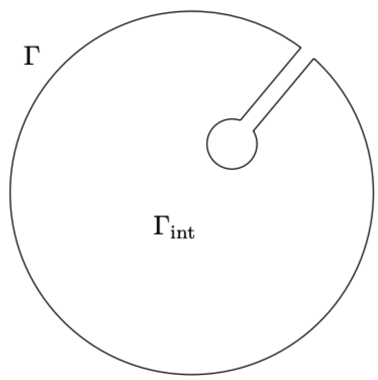
As a consequence of the Goursat's theorem, we conlude that a holomorphic function in an open disc must have a primitive in that disc. Cauchy's theorem for a disc then states that if f is holomorphic in a disc, then

$$\oint_{\gamma} f(z)dz = 0$$

for any closed curve γ in that disc. Additionally, if f is holomorphic in an open set containing the circle C (and its interior), then

$$\oint_{C} f(z)dz = 0.$$

More generally, this theorem applies whenever we can define the interior of a contour without ambiguity, and additionally draw a path between points within the contour using only vertical and horizontal segments (also within the contour). We call these horizontal and vertical paths **polygonal paths**. A **toy contour** is any closed curve where the notion of an interior is obvious, and a polygonal path can be drawn in its interior. An example of a toy contour is the following "keyhole," denoted Γ, which consists of two almost complete circles connected by a corridor.



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As before,

$$\int_\Gamma f(z)dz = 0.$$

2.3 Cauchy's Integral Formulas and Liouville's Theorem

Cauchy's Integral Formula

Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, then

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f\zeta}{\zeta - z}d\zeta, \quad \forall z \in D.$$

From this we may arrive at another property for holomorphic functions, namely their **regularity** – the fact that there exists no point in their domain where all partial derivatives are 0.

If f is holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f\zeta}{(\zeta - z)^{n+1}}d\zeta$$

for all z in the interior of C. The above equations are collectively known as **Cauchy's integral formulas**.

Cauchy's Inequalities

If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 with radius R, then

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C$ is the supremum of $|f|$ on the boundary circle C.

Moreover, f has power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Liouville's Theorem

If f is entire and bounded, then f is constant. This is a natural consequence of Cauchy's inequalities. As a corrolary, every non-constant polynomial $P(z)$ with complex coefficients has a root in $\mathbb{C}$. More specifically, every polynomial of degree $n \geq 1$ has exactly n roots in $\mathbb{C}$.

Finally, we conclude this section with a discussion of **analytic continuation**. Suppose f is holomorphic in a region Ω that vanishes on a sequences of distinct points with a limit point in Ω; then f is identically zero. More generally, if f and g are holomorphic in Ω and $f(z) = g(z)$ for all z in a non-empty open subset of Ω, then $f(z) = g(z)$ on all of Ω. If f and F are holomorphic on Ω and Ω' respectively such that $\Omega \subset \Omega'$ and both functions agree on Ω, then F is the **analytic continuation of f into the region Ω'**. Since F is uniquely determined by f, there can only be one such continuation.

3 Meromorphic Functions and the Logarithm

3.1 Zeros and Poles

A **point singularity**, also called an **isolated singularity**, of a function f is a complex number z_0 such that f is defined in a neighborhood of z_0

but not at z_0 itself. Singularities often appear because the denominator of a fraction vanishes. A complex number z_0 is a **zero** for the holomorphic function f if $f(z_0) = 0$. More specifically, analytic continuation shows the zeros of a non-trivial holomorphic function are isolated, meaning that there is a nontrivial neighborhood about each zero whose image is nonzero.

If f is holomorphic in a connected open set Ω with a zero at z_0 and does not vanish identically in Ω, there exists a neighborhood $U \subset \Omega$ about z_0, a holomorphic non-vanishing function g on U, and a positive integer n such that

$$f(z) = (z - z_0)^n g(z)$$

for $z \in U$.The number n is the **multiplicity** at z_0 (or that z_0 is a **zero of order** n). The order describes the rate at which the function vanishes.

The **deleted neighborhood** about z_0 is the r-disc about z_0 which excludes z_0. A function f defined in a deleted neighborhood about z_0 has a **pole** at z_0 if $1/f$, defined to be 0 at z_0, is holomorphic in a full neighborhood of z_0. If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a positive integer n such that

$$f(z) = (z - z_0)^{-n} h(z).$$

n is once again the **order** or **multiplicity** of the pole z_0.

Residues

If f has a pole of order n at z_0, then

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{(z - z_0)} + G(z)$$

with G a holomorphic function in a neighborhood of z_0. The sum (excluding G) is the **principal part** of f at the pole z_0 and the coefficient a_{-1} is called the **residue** of f at that pole. We write $\text{res}_{z_0} f = a_{-1}$. The residue is of special importance because all other terms in the principal part (which we denote $P(z)$) have primitives in a deleted neighborhood about z_0. This means that if we take C to be any circle centered at z_0,

$$\frac{1}{2\pi i} \oint_C P(z) dz = \text{res}_{z_0} f.$$

If f has a simple pole at z_0 then

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z).$$

A similar rule applies for higher order poles; if f has a pole of order n at z_0 then

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z).$$

3.2 The Residue Formula

Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then

$$\oint_C f(z) dz = 2\pi i \mathrm{res}_{z_0} f.$$

This can be generalized to finitely many poles, in which case if f has poles at $z_1, ..., z_n$, we have

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{z_k} f.$$

More generally, for any toy contour γ, on which f is holomorphic except for finitely many poles, we have the **residue formula:**

$$\oint_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} \mathrm{res}_{z_k} f.$$

We can use the residue formula to evaluate improper Riemann integrals of the form

$$\int_{-\infty}^{\infty} f(x) dx$$

by extending f to the complex plane and choosing a family of toy contours γ_R such that

$$\lim_{r \to \infty} \oint_{\gamma_R} f(z) dz = \int_{-\infty}^{\infty} f(x) dx.$$

As an example, take the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} dx.$$

Now consider the function

$$f(z) = \frac{1}{1+z^2},$$

which is holomorphic in $\mathbb{C}$ except for two simple poles at $\pm i$. We (somewhat arbitrarily) pick the toy contour of the upper half-circle of radius R (which, for large enough R, contains the pole at i). Then the residue of f at i is

$$\text{res}_i f = \lim_{z \to i}(z-i)\frac{1}{(z+i)(z-i)} = \frac{1}{2i};$$

then by the residue formula we have

$$\oint_{\gamma_R} f(z)dz = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

and the limit agrees with the above.

3.3 Singularities and Meromorphic Functions

Let f be a function holomorphic in an open set Ω except possibly at one point $z_0 \in \Omega$. If we can define f at z_0 at such a way that f becomes holomorphic in all of Ω, then we say z_0 is a **removable singularity**.

Riemann's Theorem on Removable Singularities

Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.

Any singularity that is not a pole or removable is an **essential singularity**, which can occur from wildly oscillating behavior near the singularity.

Casorati-Weierstrass Theorem

Suppose f is holomorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.

Picard proved a much stronger result – that under the above theorem, f takes on *every* complex value infinitely many times with at most one exception (this is called the **Great Picard theorem**).

A function f on an open set Ω is **meromorphic** if there is a sequence of of points $\{z_0, z_1, ...\}$ that has no limit points in Ω, and such that

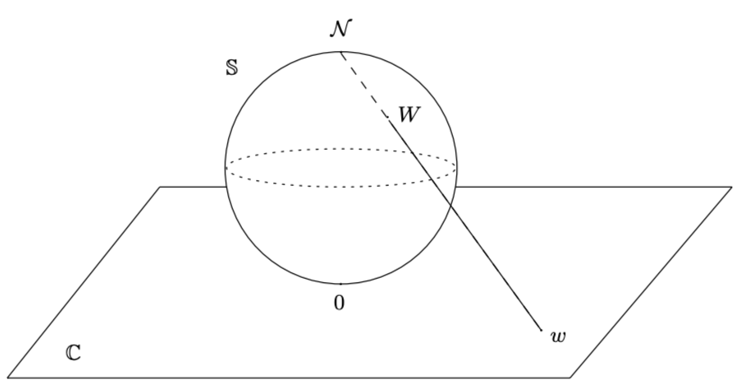
(i) the function f is holomorphic in $\Omega - \{z_0, ...\}$, and

(ii) f has poles at the points $\{z_0, ...\}$.

For functions meromorphic in the entire complex plane, we can describe its behavior at infinity using our tripartite distinction from earlier. If f is holomorphic for all large values of z, we consider $F(z) = f(1/z)$, holomorphic in a deleted neighborhood about the origin. f has a **pole at infinity** if F has a pole at the origin; f has an **essential singularity at infinity** or **removable singularity at infinity** based on the behavior of F at 0. A meromorphic function that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**; the meromorhpic functions in the extended complex plane are the rational functions.

3.3.1 The Riemann Sphere

There is a convenient way to map the unit sphere $\mathbb{S}^2$ to the complex plane; consider the unit sphere translated so that the south pole is tangent to the plane. Then consider a point W on the sphere, and a line originating from the sphere's north pole passing through W and intersecting the complex plane at a point w. The map from W to w is the **stereographic projection** of W, which forms a bijective map from the complex plane to the punctured Riemann sphere (note that the north pole itself cannot be mapped using this technique). We assign to the north pole the value of ∞, so that the construction maps the unbounded set $\mathbb{C}$ to the compact set $\mathbb{S}^2$ through the addition of a single point (hence the equivalent term **one-point compactification**).



3.4 The Argument Principle

The function $\log f(z)$ cannot be defined unambiguously on the set $f(z) \neq 0$. If we *are* to give it a definition, we define

$$\log f(z) = \log |f(z)| + i \arg f(z);$$

in either case, its derivative is $f'(z)/f(z)$ which is well-defined and the integral

$$\int_\gamma \frac{f'(z)}{f(z)} dz$$

is the change in the argument of f as z traverses γ. Assuming γ is closed, this value is determined entirely by the zeros and poles inside of γ.

The Argument Principle

Suppose f is meromorphic in an open set containing a circle C and its interior. if f has no poles and never vanishes in C, then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = (\#\text{ of zeros of } f \text{ inside } C) - (\#\text{ of poles of } f \text{ inside } C)$$

counted with multiplicities.

Rouché's Theorem

Suppose f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)|$$

then f and $f + g$ have the same number of zeros inside C.

A mapping is said to be **open** if it maps open sets to open sets. The **open mapping theorem** states that if f is holomorphic and non-constant in a region Ω, then f is open.

The **maximum** of a holomorphic function f in an open set Ω is the maximum of its absolute value $|f|$ in Ω. The **maximum modulus principle** states that if f is non-constant and holomorphic in a region Ω, then f cannot attain a maximum in Ω. Suppose Ω has compact closure $\bar{\Omega}$; so long as f is non-constant and continuous on $\bar{\Omega}$ then

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \bar{\Omega} \setminus \Omega} |f(z)|.$$

3.5 Homotopies and Simply Connected Domains

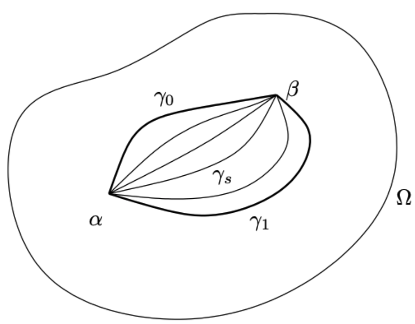
Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. So if $\gamma_0(t)$ and $\gamma_1(t)$ are parametrizations defined on $[a,b]$, we have

$$\gamma_0(a) = \gamma_1(a) = \alpha, \;\; \gamma_0(b) = \gamma_1(b) = \beta.$$

These curves are **homotopic in** Ω if for each $0 \le s \le 1$ there exists a curve $\gamma_s \subset \Omega$ parameterized by $\gamma_s(t)$ on $[a,b]$ such that for *every* s, $\gamma_s(a) = \alpha$, $\gamma_s(b) = \beta$, and for all $t \in [a,b]$,

$$\gamma_s(t)\big|_{s=0} = \gamma_0(t), \;\; \gamma_s(t)\big|_{s=1} = \gamma_1(t).$$

The functions $\gamma_s(t)$ are jointly continuous in $s \in [0,1]$ and $t \in [a,b]$. More intuitively, two curves are **homotopic** if one curve can be deformed continuously into another without leaving Ω.



If two curves γ_0 and γ_1 are homotopic in Ω and f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

A region in the complex plane is **simply connected** if any two pairs of curves with the same endpoints in Ω are homotopic in Ω. The punctured plane, $\mathbb{C} \setminus \{0\}$ is not simply connected – consider two curves with the origin between them. It is clear that it is impossible to continuously deform one curve to the other (holding the endpoints fixed) without passing over 0.

Any holomorphic function in a simply connected domain admits a primitive; from this we get a new version of Cauchy's theorem: that if f is holomorphic in the simply connected region Ω, then for any closed curve γ in Ω

$$\oint_{\gamma} f(z)dz = 0.$$

The above fact that the punctured plane is *not* simply connected can be shown by realizing the integral of $1/z$ over the unit circle in $\mathbb{C} \setminus \{0\}$ is $2\pi i$, and not 0.

3.6 The Complex Logarithm

If $z = re^{i\theta}$ and we wish the logarithm to be the inverse of the exponential, then we determine the logarithm to be

$$\log z = \log r + i\theta.$$

The issue here is that θ is only uniquely determined up to 2π. We can therefore only define the complex logarithm *locally*, and not *globally*. For example, suppose z starts at 1, winds around the origin once, and returns to 1. $\log z$ would *not* return to its original value, but instead be off by a multiple of $2\pi i$. The logarithm in this way is not **single-valued**. To make this function single-valued, we must restrict the set on which it is defined; this is called the **branch** of the logarithm.

Branches

Suppose that Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that

(i) F is holomorphic in Ω;

(ii) $e^{F(z)} = z$ for all $z \in \Omega$;

(iii) $F(r) = \log r$ whenever $r \in \mathbb{R}$ and near 1.

In this way, the branch $\log_\Omega(z)$ is an extension of the logarithm for positive numbers. In the slit plane $\Omega = \mathbb{C} \setminus \{(-\infty, 0]\}$, we have the **principal branch of the logarithm**

$$\mathrm{Log} z = \log r + i\theta$$

with $|\theta| < \pi$.

Every non-zero complex number w can be written as $w = e^z$. More specifically, we have the following, which discusses the existence of $\log f(z)$ whenever f does not vanish. If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that

$$f(z) = e^{g(z)}.$$

The function g can be denoted by $\log f(z)$, and uniquely determines a branch of that logarithm.

3.7 Fourier Series and Harmonic Functions

Suppose f is holomorphic in a disc $D_R(z_0)$, so that f has power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges in that disc. The coefficients of the power expansion series of f are given by:

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

for all $n \geq 0$ and $0 < r < R$. Moreover:

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

when $n < 0$. Since $a_0 = f(z_0)$, we obtain the following property:

Mean-Value Property

If f is holomorphic in a disc $D_R(z_0)$ then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

for any $0 < r < R$. As a consequence, if $u = \mathfrak{Re}(f)$, then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

for any $0 < r < R$. u is harmonic whenever f is holomorphic, and this property is shared by *every* harmonic function in the disc $D_R(z_0)$.

4 The Fourier Transform

If f is a function satisfying appropriate conditions, its **Fourier transform** is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx, \quad \xi \in \mathbb{R}.$$

Its counterpart, the **Fourier inversion**, also holds:

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad x \in \mathbb{R}.$$

The Fourier transform performs a basic role in analysis; here, we go further and illustrate the intimate connection between the Fourier transform and complex analysis. For a function f, initially defined on the real line, the possibility of *extending* that function to a holomorphic one is related to the extreme decay at infinity of its Fourier transform.

Assume f can be analytically continued in a strip containing the real axis, such that the integral defining the Fourier transform $\hat{f}$ converges. For convergence, $\hat{f}$ must decay exponentially at infinity, an elegant consequence of contour integration. Secondly, we may then ask for the conditions on f such that its Fourier transform has bounded support, say on $[-M, M]$. This question can only be resolved in terms of the holomorphic properties of f. This condition is given in section 4.3.

4.1 The Class $\mathfrak{F}$

In other courses, you may have studied the decay conditions of the Fourier transform, and the class of functions under "moderate decay" (e.g. the Poisson kernel for solving the Dirichlet problem for the steady-state heat equation in the upper half-plane). Here, we determine a class of functions that is large enough to contain many more important applications.

The Class $\mathfrak{F}$

For each $a > 0$, we denote by $\mathfrak{F}_a$ the class of all functions f such that:

(ii) There exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \text{ For all } x \in \mathbb{R}, \; |y| < a.$$

(i) The function f is holomorphic in

$$S_a = \{z \in \mathbb{C} : |\mathfrak{Im}(z)| < a\}.$$

The class $\mathfrak{F}$ is the class of *all* functions that belong to some $\mathfrak{F}_a$ for any a.

4.2 Action of the Transform on $\mathfrak{F}$

Here we present three essential theorems that are all rooted in contour integration.

(i) If f belongs to the class $\mathfrak{F}_a$ for some $a > 0$, then
$$|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}$$
for any $0 \leq b < a$.

(ii) (**Fourier Inversion Theorem**) If $f \in \mathfrak{F}$, then the Fourier inversion holds, namely
$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i x\xi}d\xi \ \text{ for all } x \in \mathbb{R}$$

(iii) (**Poisson Summation Formula**) If $f \in \mathfrak{F}$, then
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

The remainder of this section discusses several important identities that are critical consequences of the aforementioned identities.

First, recall that the function $e^{-\pi x^2}$ is its own Fourier transform:
$$\int_{\mathbb{R}} e^{-\pi x^2}e^{-2\pi i x\xi}dx = e^{-\pi\xi^2}.$$
For fixed values of $t > 0$ and $a \in \mathbb{R}$, the change of variables $x \mapsto t^{1/2}(x + a)$ above shows that the Fourier transform is then
$$f(x) = e^{-\pi t(x+a)^2} \iff \hat{f}(\xi) = t^{-1/2}e^{-\pi\xi^2/t}e^{2\pi i a\xi}.$$
Then using the Poisson summation formula yields the relation
$$\sum_{n\in\mathbb{Z}} e^{-\pi t(n+a)^2} = \sum_{n\in\mathbb{Z}} t^{-1/2}e^{-\pi n^2/t}e^{2\pi i n a}.$$
In the above identity, the assignment $a = 0$ yields the transformation law for a version of the **theta function**. If we define, for $t > 0$, the series
$$\vartheta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t},$$
then by the above relation we may write
$$\vartheta(t) = t^{-1/2}\vartheta(1/t),$$
a key result that will be used in the next chapter to derive the functional equation for the Riemann-Zeta function and its analytic continuation.

4.3 Paley-Wiener and Phragmén-Lindelöf

In this section we reverse our thinking – we do not suppose *any* analyticity of f, but we *do* assume the validity of the inversion formula.

Suppose $\hat{f}$ satisfies the decay condition $|\hat{f}(\xi)| \leq Ae^{-2\pi a|\xi|}$ for some constants $a, A > 0$. Then $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ holomorphic in the strip

$$S_b = \{z \in \mathbb{C} : |\mathfrak{Im}(z)| < b\}$$

for any $0 < b < a$. As a corollary, if $\hat{f}(\xi) = O(e^{-2\pi a|\xi|})$ for some $a > 0$, and f vanishes in a non-empty open interval, then $f = 0$.

Paley-Wiener Theorem

Suppose f is continuous and of moderate decrease on $\mathbb{R}$. Then f has an extension to the complex plane that is entire with

$$|f(z) \leq Ae^{2\pi M|z|}$$

for some $A > 0$, if and only if $\hat{f}$ is supported on the bounded interval $[-M, M]$.

Phragmén-Lindelöf Principle for Complex Sectors

Suppose F is a holomorphic function in the sector

$$S = \{z : -\pi/4 < \arg z < \pi/4\}$$

that is continuous on the closure of S. Assume $|F(z)| \leq 1$ on the boundary of the sector, and that there are constants $C, c > 0$ such that $|F(z)| < Ce^{c|z|}$ for all z in the sector. Then

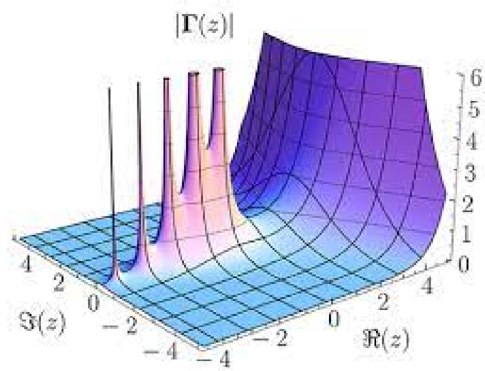
$$|F(z)| < 1 \text{ for all } z \in S.$$

In other words, if F is bounded by 1 on the boundary of a sector S and has an at most reasonable amount of growth, then F is bounded everywhere by 1. The growth condition is required; the function e^z, for example, is unbounded on the real line. This theorem may be thought of as a kind of "generalization" of the maximum modulus principle.

5 The Gamma and Zeta Functions

The gamma and zeta functions are among the most important nonelementary functions in mathematics. The function $1/\Gamma(s)$ is the simplest entire function, with zeros at exactly $0, -1, -2, \ldots$. The zeta function ζ has a huge role in the analytic theory of numbers, and is intimately connected with the prime numbers. We also discuss the theta and xi functions, which are variants of the zeta function that add other interesting symmetries.

5.1 The Gamma Function



For $s > 0$, the **gamma function** Γ is

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

This integral converges for $s > 0$ due to the integrability of t^{s-1} near the origin and the exponential decay of e^{-t} as t tends to infinity. The gamma function extends to an analytic function in the half-plane $\mathfrak{Re}(s) > 0$ and is still given by the above integral formula.

5.1.1 Analytic Continuation

The integral defining Γ is not absolutely convergent for other values of s; however, we may use analytic continuation to find a meromorphic function on all of $\mathbb{C}$ that equals Γ in $\mathfrak{Re}(s) > 0$.

To this end, through integration by parts we may determine the following relation, that if $\mathfrak{Re}(s) > 0$, then

$$\Gamma(s+1) = s\Gamma(s)$$

and as a consequence $\Gamma(s) = (n-1)!$ for all non-negative integers n. From this, Γ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose singularities are exclusively simple poles at the negative integers $s = 0, -1, ...$ The residue of Γ at $s = -n$ is $\text{res}_{-n}(\Gamma) = (-1)^n/n!$.

5.1.2 Properties of Γ

Γ is symmetric about the line $\mathfrak{Re}(s) = 1/2$. For all $s \in \mathbb{C}$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

$\Gamma(1-s)$ has simple poles at all positive integers, so $\Gamma(s)\Gamma(1-s)$ is meromorphic with simple poles at *every* integer, a property shared by $\pi/\sin \pi s$. This can be proven using the lemma that, for $0 < a < 1$,

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \frac{\pi}{\sin \pi a}.$$

We continue our study with the study of the reciprocal of the gamma function, which is entire. The function Γ has the following properties:

(i) $1/\Gamma$ is an entire function of s with simple zeros at exclusively the non-positive integers

(ii) $1/\Gamma(s)$ has growth

$$\left|\frac{1}{\Gamma(s)}\right| \leq c_1 \exp\left\{c_2 |s| \log |s|\right\},$$

meaning that for every $\varepsilon > 0$ there exists a bound $c(\varepsilon)$ such that

$$\left|\frac{1}{\Gamma(s)}\right| \leq c_1 \exp\left\{c_2 |s|^{1+e}\right\}.$$

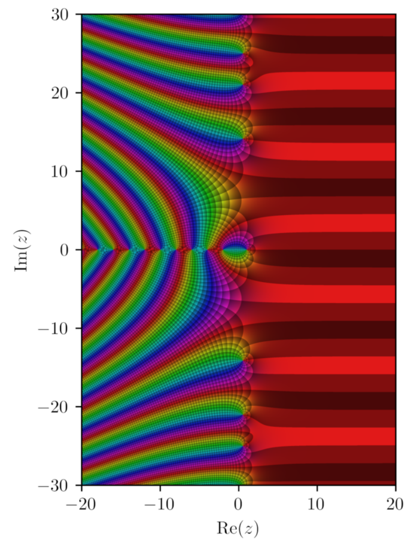
The growth condition on $1/\Gamma$ leads to the product formula for $1/\Gamma$:

$$\frac{1}{\Gamma(s)} = e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n},$$

where γ is the real number known as the **Euler-Mascheroni constant** and is given by

$$\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$

5.2 The Riemann Zeta Function



The **Riemann zeta function** is given by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As with γ, ζ can be extended to the complex plane via a simple extension to the complex half plane. The series defining $\zeta(s)$ converges for $\mathfrak{Re}(s) > 1$ and the function ζ is holomorphic in this half-plane. This convergence can be shown by analyzing the quantity $|n^{-s}|$ given that $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$.

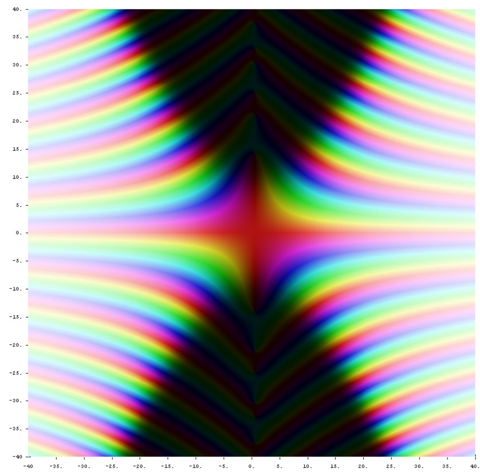
The analytic continuation of ζ to a meromorphic function is more subtle than that of Γ. To this end, we make use of the **theta function** (mentioned in the previous chapter), defined for $t > 0$ as

$$\vartheta(t) = \sum_{\mathbb{R}} e^{-\pi n^2 t},$$

with functional equation

$$\vartheta(t) = t^{-1/2} \vartheta(1/t).$$

5.2.1 The Riemann Xi Function



The **Riemann xi function** is a modification of the Riemann zeta function to make it more symmetric. It is a relationship between the gamma, zeta, and theta functions as follows:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}[\vartheta(u) - 1]du.$$

$\xi(s)$ is holomorphic for $\mathfrak{Re}(s) > 1$ with an analytic continuation to $\mathbb{C}$ as a meromorphic function with simple poles at $s = 0$ and $s = 1$. $\xi(s)$ has the property that $\xi(s) = \xi(1-s)$. This definition of ξ directly yields the analytic continuation and functional form of the Riemann zeta function; the zeta function has a meromorphic continuation into the complex plane whose only singularity is a simple pole at $s = 1$. The meromorphic continuation in question is then

$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}.$$

A more elementary approach to analytic continuity helps clarify the growth properties of ζ near the line $\mathfrak{Re}(s) = 1$; it helps give clarity to the nature of the zeros of ζ, and the famous unsolved **Riemann hypothesis** – that all the zeros of ζ in the **critical strip** between $\mathfrak{Re}(s) = 0$ and $\mathfrak{Re}(s) = 1$ lie on the line $\mathfrak{Re}(s) = 1/2$. The idea compares the sum $\sum_{n=1}^\infty n^{-s}$ with $\int_1^\infty x^{-s}dx$. We propose that there exists a sequence of functions $\{\delta_n(s)\}_{n=1}^\infty$ that satisfy $|\delta_n(s)| \leq |s|/n^{\sigma+1}$ for $s = \sigma + it$, such that

$$\sum_{1\leq n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\leq n<N} \delta_n(s).$$

As a consequence,

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

for $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holomorphic in $\mathfrak{Re}(s) > 0$. This idea can be iterated upon to yield a continuation into not just the real-positive half-plane, but to all of $\mathbb{C}$.